Lecture 21 04/15/2019

Radiation (Cont'd)

Now, we consider the $l=1$ term in the expansion for the vector potential.

For $r>r'$, we have:

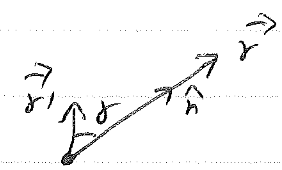
$$\vec{A}(\vec{x}) = \mu_0 ik\, h_1^{(1)}(kr) \int \vec{J}(\vec{x}')\, j_1(kr') \sum_{m=-1}^{+1} Y_{1m}(\theta,\phi) Y_{1m}^*(\theta',\phi')$$

In the long wavelength approximation, where $kr' \ll 1$, we have:

$$j_1(kr') \simeq \frac{kr'}{3}$$

Also, note that:

$$\sum_{m=-1}^{+1} Y_{1m}(\theta,\phi) Y_{1m}^*(\theta',\phi') = \frac{3}{4\pi} P_1(\cos\gamma)$$



Thus:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \int \vec{J}(\vec{x}')\, (\hat{n}\cdot\vec{x}')\, d^3x'$$

We can write the integral as:

$$n_j \hat{e}_i \int J_i(\vec{x}')\, x'_j\, d^3x' = n_j \hat{e}_i \left\{ \frac{1}{2} \int \left[x'_j J_i(\vec{x}') - x'_i J_j(\vec{x}') \right] d^3x' + \right.$$

$$\left. \frac{1}{2} \int \left[x'_j J_i(\vec{x}') + x'_i J_j(\vec{x}') \right] d^3x' \right\}$$

The first integral is:

$$n_j \hat{e}_i \propto \frac{1}{2} \varepsilon_{ijk} \int \left[\vec{x}' \times \vec{J}_{(\vec{x}')}\right]_k d^3x' = \vec{m} \times \hat{n}$$

Where $\vec{m}$ is the magnetic dipole moment of the source (the physical magnetic dipole moment is $Re(\vec{m}e^{-i\omega t})$). The magnetic dipole contribution to the vector potential is:

$$\vec{A}_m(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right) \vec{m} \times \hat{n} \underset{\substack{kr \gg 1 \\ \text{limit}}}{\simeq} ik \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m})$$

Then (in the far-field region $kr \gg 1$):

$$\vec{B}_m(\vec{x}) = \vec{\nabla} \times \vec{A}_m(\vec{x}) \simeq \frac{k^2 \mu_0}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m}) \times \hat{n}$$

$$\vec{E}_m(\vec{x}) = \frac{-1}{\mu_0 \varepsilon_0 i\omega} \vec{\nabla} \times \vec{B}_m(\vec{x}) \simeq \frac{\mu_0 c k^2}{4\pi} \frac{e^{ikr}}{r} \left((\hat{n} \times \vec{m}) \times \hat{n}\right) \times \hat{n} \underset{\hat{n}\cdot\hat{n}=1 \text{ used}}{=} \frac{-\mu_0 c k^2}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m})$$

This results in:

$$\frac{dP_m}{d\Omega} = \frac{1}{2\mu_0} Re\left[\vec{E}_m(\vec{x}) \times \vec{B}_m^*(\vec{x})\right] \cdot \hat{n}\, r^2 \underset{(\hat{n} \times \vec{m}) \cdot \hat{n} = 0 \text{ used}}{=} \frac{\mu_0 c k^4}{32\pi^2} |\hat{n} \times \vec{m}|^2$$

We see that all considerations of electric dipole radiation apply here upon making $\vec{P} \to \frac{\vec{m}}{c}$, $\vec{E} \to c\vec{B}$, $\vec{B} \to -\frac{\vec{E}}{c}$ replacements. Note that the polarization of radiation is determined by the components of $\hat{n} \times \vec{m}$.

The second integral on the right-hand side of the expression at the end of page 1 can be written as:

$$\frac{1}{2}\int [x'_j J_i + x'_i J_j]\, d^3x' = \frac{1}{2}\int \Big[\partial'_k (x'_j x'_i J_k) - x'_j x'_i \underbrace{\partial'_k J_k}_{\vec{\nabla}' \cdot \vec{J}}\Big] d^3x'$$

The first term on the right-hand side gives rise to a surface integral that, due to the source being localized, vanishes. After using $\vec{\nabla}' \cdot \vec{J}(\vec{x}') = i\omega \rho(\vec{x}')$, as shown before, we find:

$$\vec{A}_q(\vec{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\left(-\frac{i\omega}{2}\right)\left[n_j \hat{e}_i \int x'_j x'_i \rho(\vec{x}')\, d^3x'\right]$$

But:

$$\int x'_j x'_i \rho(\vec{x}')\, d^3x' = \frac{1}{3}\int [3x'_i x'_j - r'^2 \delta_{ij}]\, \rho(\vec{x}')\, d^3x' + \frac{1}{3}\delta_{ij}\int r'^2 \rho(\vec{x}')\, d^3x'$$

$$= \frac{1}{3} Q_{ij} + \frac{1}{3}\delta_{ij}\int r'^2 \rho(\vec{x}')\, d^3x'$$

Here $Q_{ij} \equiv \int [3x'_i x'_j - r'^2 \delta_{ij}]\, \rho(\vec{x}')\, d^3x'$ is the ij-th component of the electric quadrupole tensor of the source (hence the subscript "q" in $\vec{A}_q$). In the far-field region, therefore, we have:

$$\vec{A}_q(\vec{x}) \simeq \frac{-\mu_0 \omega k}{24\pi} \frac{e^{ikr}}{r} \left[(Q_{ij} n_j)\hat{e}_i + \hat{n} \int \rho(\vec{x}')\, r'^2\, d^3x' \right]$$

And:

$$\vec{B}_q(\vec{x}) = \vec{\nabla} \times \vec{A}_q(\vec{x}) \simeq \frac{-i\mu_0 c k^3}{24\pi} \frac{e^{ikr}}{r}\, \hat{n} \times \vec{q}(\hat{n}) \quad \left(q_i(\hat{n}) \equiv \sum_{j=1}^{3} Q_{ij} n_j \right)$$

$$E_q(\vec{x}) = \frac{i}{\mu_0 \epsilon_0 \omega} \vec{\nabla} \times \vec{B}_q(\vec{x}) \simeq \frac{ick^4}{\epsilon_0 \omega} \frac{e^{ikr}}{24\pi r}\, \hat{n} \times (\hat{n} \times \vec{q}(\hat{n})) = \frac{ik^3}{24\pi\epsilon_0} \frac{e^{ikr}}{r}\, \hat{n} \times (\hat{n} \times \vec{q}(\hat{n}))$$

Thus:

$$\frac{dP_q}{d\Omega} \simeq \frac{1}{2\mu_0} \mathrm{Re}\left[\vec{E}_q(\vec{x}) \times \vec{B}_q^*(\vec{x}) \right] \cdot \hat{n}\, r^2 \propto k^6 \left[|\vec{q}(\hat{n})|^2 - |\hat{n} \cdot \vec{q}(\hat{n})|^2 \right]$$

$$|\hat{n} \times \vec{q}(\hat{n})|^2 = (\hat{n} \times \vec{q}(\hat{n})) \cdot (\hat{n} \times \vec{q}^*(\hat{n})) = \vec{q}(\hat{n}) \cdot \vec{q}^*(\hat{n}) - |\hat{n} \cdot \vec{q}(\hat{n})|^2$$

In comparison with the $l=0$ term (i.e., electric dipole radiation), power in the electric quadrupole (and possibly magnetic dipole) scales as $k^6 d^4$ as opposed to $k^4 d^2$ ("d" being the typical size of the source). This implies that in the long wavelength limit, when $kd \ll 1$, terms with higher l are suppressed in power of kd, thereby making the expansion a useful approximation.

We note that in cases that the electric dipole moment contribution is absent, we must consider the magnetic dipole and electric quadrupole contributions as the first non-vanishing terms. An example of such a situation is two equal charges at diametrically opposite points in uniform rotational motion on a circle.

Finally, let us comment on the general situation when the dimension of the source is not necessarily small compared to the wave length of radiation. In this case, in the radiation zone $r \gg r', \lambda$, we have:

$$|\vec{x}-\vec{x}'| = \sqrt{r^2+r'^2-2\vec{x}\cdot\vec{x}'} = r\left(1-\frac{2\vec{x}\cdot\vec{x}'}{r^2}+\frac{r'^2}{r^2}\right)^{\frac{1}{2}} = r-\hat{n}\cdot\vec{x}'+O\left(\frac{r'^2}{r^2}\right)$$

Thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, e^{ik|\vec{x}-\vec{x}'|}\, d^3x' \simeq \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\, \frac{e^{ik(r-\hat{n}\cdot\vec{x}')}}{r-\hat{n}\cdot\vec{x}'}\, d^3x'$$

Now, we can neglect $\hat{n}\cdot\vec{x}'$ in the denominator as compared with r. However, we cannot do the same in the exponent of the numerator as $k\hat{n}\cdot\vec{x}'$ may still be a large phase. Hence:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}\, d^3x'$$

This is just the Fourier transform of the current density! As for the $\vec{B}$ and $\vec{E}$ fields, we have:

$$\vec{B} = \vec{\nabla}\times\vec{A} \simeq ik\hat{n}\times\vec{A}$$

$$\vec{E} = \frac{ic^2}{\omega}\vec{\nabla}\times\vec{B} \simeq \frac{ic^2}{\omega}ik\hat{n}\times\vec{B} \simeq -ik\hat{n}\times(\hat{n}\times\vec{A})$$